**Final Examination, 15 December 1998**
**SM311O (Fall 1998)**

*The following formulas may be useful to you:*

$$a) \oint_C \mathbf{v}\cdot d\mathbf{r} = \int\int_S \nabla\times\mathbf{v}\cdot d\mathbf{A},$$

$$b)\ \rho(\frac{\partial \mathbf{v}}{\partial t} + \nabla\mathbf{v}\cdot\mathbf{v}) = -\nabla p + \mu\Delta\mathbf{v} + \rho\mathbf{F}, \qquad \text{div }\mathbf{v} = 0.$$

$$c) \quad -fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + A_V\frac{\partial^2 u}{\partial z^2}, \quad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + A_V\frac{\partial^2 v}{\partial z^2}, \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$

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1. (a) Let $\mathbf{v} = \langle x^2 z, y\sin(xy), e^{-xz}\rangle$. Find the divergence of $\mathbf{v}$.
   (b) $\mathbf{v} = \langle \frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}}, 0\rangle$. Find the curl of $\mathbf{v}$.
   (c) Let $f(x,y) = e^{2x}\sin 3\pi y$. Find the direction of steepest descent at $P = (0, \frac{1}{4})$.

2. Verify by direct differentiation if
   (a) $u(z) = e^z\cos z$ is a solution of $u'''' + a^2 u = 0$ for any $a$.
   (b) $u(x,y) = \sin 3x \sin 4y$ is an eigenfunction of the Laplace operator $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$. What is the eigenvalue?

3. Parametrize the following curves and surfaces:
   (a) The plane passing through the points $(1,1,0)$, $(0,3,2)$, and $(1,0,5)$.
   (b) The curve $x^2 + 3y^2 = 2$.
   (c) The upper hemisphere of radius 2 centered at $(1,-1,1)$.

4. (a) The function $\phi(x,y,z) = ax^2y^2 + by^2z^2 + cz^2x^2 - x$ is the potential for a velocity vector field $\mathbf{v}$. Determine the values of $a$, $b$, and $c$ so that the velocity of the particle located at $(1,-2,3)$ is zero.
   (b) The function $\psi(x,y) = ax^2 - xy + by^2$ is the stream function of a velocity field $\mathbf{v}$. Find $a$ and $b$ so that the velocity of the particle located at $(-1,4)$ is perpendicular to $\langle 1,1\rangle$.

5. (a) Consider the velocity vector field $\mathbf{v} = \langle x^2 - y^2, -2xy + x^3\rangle$. Does this vector field have a stream function? If yes, find it.
   (b) Let $u$ and $v$ be the velocities in a geostrophic flow with $A_V = 0$. What is the geometric relationship between the isobars and the particle paths of typical fluid particles?

6. A flow is called barotropic if the pressure field $p$ is a function of the density $\rho$, that is, $p = f(\rho)$ for some function $f$.
   (a) Compute $\nabla p$ in terms of $\nabla \rho$. What can you conclude about $\nabla p \times \nabla \rho$ in a barotropic flow? Why? What does this result say about the isobars and isopycnals of the flow? Why?
   (b) Let $f(x) = x^2 + 3x$. Suppose that the density at $P = (1, 2, 3)$ is 1.003. Furthermore, suppose that the pressure gradient at $P$ is $\langle 3, 4, 2 \rangle$. What is the density gradient at $P$?

7. (a) Let $p$ and $\rho$ be two arbitrary smooth functions of $x$, $y$, and $z$. Use direct differentiation and prove the identity
$$\nabla \times \left( \frac{1}{\rho} \nabla p \right) = -\frac{1}{\rho^2} (\nabla \rho \times \nabla p).$$
   (b) Use the above identity, the conclusion in 6(a), and the Stokes theorem to compute the line integral $\oint_C \frac{1}{\rho} \nabla p \cdot d\mathbf{r}$ where $C$ is a closed curve and $p$ and $\rho$ are the pressure and density of a barotropic flow.

8. Consider the following heat conduction problem:
$$u_t = 5u_{xx}, \qquad u(0,t) = u(2,t) = 0, \qquad u(x,0) = 1.3 \sin \frac{\pi x}{2} + 3.4 \sin 2\pi x.$$
   (a) Describe a physical model for which the above BVP makes sense.
   (b) Assuming that $u$ is the temperature, find the units of the physical quantity whose value is 5 in the heat equation.
   (c) Use separation of variables and find the solution to this problem.
   (d) Use the first nonzero term of the above solution and estimate how long it takes for the temperature at $x = 1$ to reach 50 per cent of its original value.

9. Let $\mathbf{v} = \langle y^2, 2xy \rangle$ be the velocity field of a fluid.
   (a) Compute the vorticity of the flow. Is the flow irrotational anywhere in the $xy$-plane?
   (b) Compute the acceleration $\mathbf{a}$ of this flow. Does $\mathbf{a}$ have a potential $p$? If yes, find it.

10. Consider the viscous geostrophic equations listed on the previous page. Assuming that $u$ and $v$ are only functions of $z$, that $\rho$, $f$, and $A_V$ are constants, and that $\lim_{z\to\infty} u(z) = U$, a constant, and $\lim_{z\to\infty} v(z) = 0$,
    (a) Prove that $\nabla p$ must be a constant vector.
    (b) Find the ODE that $u$ must satisfy.